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ON THREE DIMENSIONAL DETERMINANTS.*

By E. R. Hedrick.†

1. Introduction.

The extension of the idea of ordinary (or plane) determinants to arrays in three dimensions was first made by de Gasparis‡ in 1861, who defined a determinant of three dimensions, and gave some simple theorems, including the multiplication theorem. He was followed by Padova,§ Armenanti,‖ Garbieri,¶ Zehfuss,** and Dahlander;†† of whom the latter two probably worked independently of de Gasparis. None of these papers contain much new work, except that of Zehfuss, who gives the expressions for some elementary invariants. Later, several more important papers appeared, by Lloyd Tanner,‡‡ Scott,§§ Gegenbauer,‖‖ and von Escherich.¶¶ Of these, the papers of Scott and Gegenbauer are most important. Scott deals mainly with determinants of special form; and Gegenbauer studies determinants of special form, and the application of the work to the invariant theory and to complex number systems. Both Gegenbauer and von Escherich have complicated their work greatly, and it would seem needlessly, by working in m dimensions at once, in place of 3 dimensions. For while essentially new ideas are introduced in passing from two to three dimensions, the generalization past that point is perfectly obvious and presents nothing new.*

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* This paper was read before the American Mathematical Society at the meeting of Feb. 25, 1899.

† A considerable portion of the material for this paper was prepared in collaboration with Mr. W. D. Cairns, now of Oberlin College, O.; but he should not be held responsible for any statements made in it.

‡ A brochure entitled "Sur les déterminants dont les éléments ont plusieurs indices." [I have not seen this brochure.] Also: *Rend. dell' accad. Napoli.*, VII, p. 118, 1868.

§ *Batt. Gior. di Mat.*, VI, p. 182.

‖ *Ibid.*, VI, p. 175.

¶ *Ibid.*, XV, p. 89.

** *Ueber eine Erweiterung des Begriffes der Determinanten.* Frankfurt a -M., 1868.

†† *Öfvers. af K. Vet. Akad. Förh.* Stockholm, 1863, p. 295.

‡‡ *Proc. Lond. Math. Soc.*, X, 1879, p. 167.

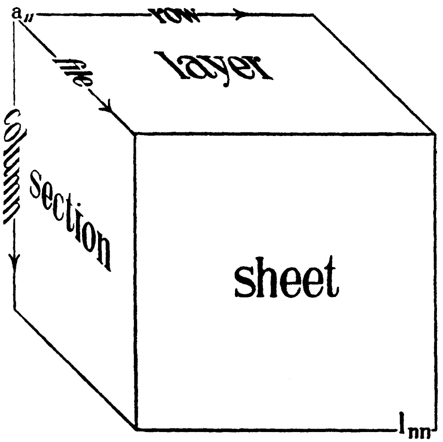
§§ *Ibid.*, XI, 1880, p. 17; *ibid.*, XIII, 1882, p. 33.

‖‖ *Wien. Akad. Denkschriften*, Bd. 43, 1881, p. 17; *ibid.*, Bd. 46, 1883, p. 291; *ibid.*, Bd. 50, 1885, p. 145; *ibid.*, Bd. 55, 1889, p. 39; *ibid.*, Bd. 57, 1890, p. 739. *Wien. Akad. Sitzungsberichte*, Bd. 101,2ᴬ, 1892, p. 425.

¶¶ *Wien. Akad. Denkschriften*, Bd. 43, 1881, p. 1.

In this paper, it is proposed to take a somewhat new point of view, in basing the work on a new definition of the determinant. The work will be done for 3 dimensions only, thus securing the aid of a geometric picture. It is hoped that the simplicity thus gained will more than compensate for the slight labor of extending the theorems to the case of m dimensions. In the course of the paper several results will be stated, which are believed to be new.

2. Definition.



Let us think of n^3 elements placed in an array in a cube in a manner similar to the arrangement of n^2 elements in a square to form an ordinary determinant. There will be three mutually perpendicular lines, which we will call *rows*, *columns*, and *files*, respectively, if they are horizontal, vertical, perpendicular to the plane of the paper. There will be three mutually perpendicular planes, which we will call *layers*, *sections*, and *sheets*, respectively, if they are perpendicular to the *columns*, *rows*, *files*.

We now define† the value of the cubic‡ determinant to be the algebraic sum obtained by adding all possible determinants formed by taking any set of n columns, no two of which lie in the same section or sheet, and constructing a determinant of these columns, placing them in the order in which their respective sheets occur in the cubic determinant; and affixing the sign + or − according as the number of inversions in the order of the sections from which the columns came, is even or odd. There are $n!$ ordinary determinants in the expansion of a cubic determinant of order n. Thus for a determinant of the third order:

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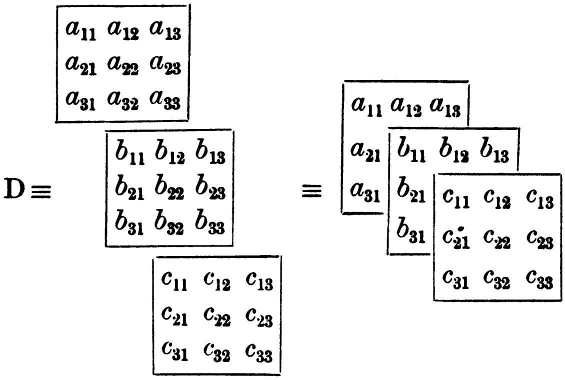
* Reference to the subject is made in the following, also. It is believed that these make the list fairly complete. (1) Waelsch, *Wien. Monatshefte*, IX, p. 213. (2) *Encyklopädie der Math. Wiss.*, Bd. 1, p. 45. (3) Günther, *Lehrbuch der Determinantentheorie*, p. 186 *et seq.* (4) Scott, *Theory of Determinants*, pp. 89–98. (5) Mansion, *Elèments de la theorie des dèterminants*, Ed. 1883, p. 22 (note).

† This definition is strictly analogous to the definition of ordinary determinants in terms of products.

‡ So, in general, we should define a determinant of m dimensions (or of *class* m) to be equal to the sum of $n!$ determinants of class $(m - 1)$.



$$\equiv \begin{vmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{vmatrix} + \begin{vmatrix} a_{12} & b_{13} & c_{11} \\ a_{22} & b_{23} & c_{21} \\ a_{32} & b_{33} & c_{31} \end{vmatrix} + \begin{vmatrix} a_{13} & b_{11} & c_{12} \\ a_{23} & b_{21} & c_{22} \\ a_{33} & b_{31} & c_{32} \end{vmatrix} - \begin{vmatrix} a_{11} & b_{13} & c_{12} \\ a_{21} & b_{23} & c_{22} \\ a_{31} & b_{33} & c_{32} \end{vmatrix} - \begin{vmatrix} a_{12} & b_{11} & c_{13} \\ a_{22} & b_{21} & c_{23} \\ a_{32} & b_{31} & c_{33} \end{vmatrix} - \begin{vmatrix} a_{13} & b_{12} & c_{11} \\ a_{23} & b_{22} & c_{21} \\ a_{33} & b_{32} & c_{31} \end{vmatrix}$$

It is seen from this expansion that the sign of any determinant in the expansion will be + or −, according as the number of inversions in it of the last subscript (in the above notation) is even or odd.

Also, the cubic determinant is seen to be equal to the sum of the *products* formed by taking the product of any n elements, no two of which lie in the same layer, section, or sheet and affixing to it the sign + or −, according as the sum of the number of inversions of the two subscripts, when the letters are arranged alphabetically, is even or odd.* There are $(n!)^2$ terms in this expansion.

It now follows that in the above definition we may interchange the words *layer* and *section*, and simultaneously the words *row* and *column*, for the final developments will be precisely as before. Thus, in the previous example:

$$D \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{31} & b_{32} & b_{33} \\ c_{21} & c_{22} & c_{23} \end{vmatrix} + \textit{four more similar determinants.}$$

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* This has been taken as a *definition* by other writers, and the above definition deduced as a theorem.

If we wish to develop by *files*, we shall find no development into ordinary determinants.* If we define a function, however, which is formed exactly like an ordinary determinant, except that each of its terms is positive:

$$\left\{\begin{matrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ q_1 & q_2 & \dots & q_n \end{matrix}\right\} \equiv \sum (a_1\, b_2 \; \dots \; q_n)$$

we see at once that we may develop by files in terms of these *Scott's functions*,† as I shall call them. Form all the Scott's functions possible by taking for the rows any n files of the cubic determinant, no two of which lie in the same layer or section; affix to each the known sign of its principal diagonal; — the sum of these functions is the cubic determinant. For, it is seen that every term resulting from the complete expansion of the cubic determinant occurs (except perhaps for sign) in the complete expansion of this sum of Scott's functions. Moreover, the signs will be correct. For the sign of all terms resulting from the expansion of any one of the above Scott's functions, will be the same in the complete development of the cubic determinant.

3. ELEMENTARY THEOREMS.

In the following theorems, the proofs will be indicated only when some point is particularly interesting or difficult. In general the method of proof is to develop the cubic determinant into ordinary determinants, when the truth is at once seen. No effort has been made to render this list complete, for too many theorems suggest themselves at once by analogy from two dimensional determinants; and only those will be found here which are to be used in this paper. The first writer who has stated a theorem is indicated in parentheses at its close. The proofs in this paper are generally different from those given by others, as the theorems themselves, in all cases, were found independently. The writer regrets not having seen de Gasparis' first paper.

(1). In any cubic determinant, the sections may be interchanged with the layers, in order, without altering the value of the determinant. But the sheets cannot so be interchanged either with the sections or with the layers. (Padova.) Proof by development into plane determinants, by Sec. 2.

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* This lack of symmetry is essential, and is noticeable throughout the work. It is due to the fact that the number of inversions of three series of numbers when grouped into sets of three each, one from each set, is not constant when the order of arrangement of the sets is changed.

† Such a function has been defined and used by F. R. Scott: *Proc. Lond. Math. Soc.*, Vol. XIII, p. 36.

(2). In any cubic determinant, if two adjacent sections be interchanged, the determinant changes sign. The same thing is true for layers. But any two sheets may be interchanged without any alteration of the determinant. (Dahlander.) Proof by development into plane determinants.

(3). If each of the elements of any layer (or section) be equal to the corresponding element of any other layer (or section), the determinant vanishes. But two (or any number) of the sheets may be identical, and the determinant will not necessarily vanish. (Dahlander.) Proof by development into plane determinants.

(4) If each of the elements of any plane (layer, section, or sheet) be multiplied by any quantity, the value of the determinant is multiplied by that quantity. (Gegenbauer, 1881.) Proof by development into plane determinants by Sec. 2; or into Scott's functions if we are dealing with sheets (and hence files). It is necessary to notice that if any row of a Scott's function be multiplied by any quantity, the function is multiplied by that quantity. This comes directly from the definition.

(5). The value of a cubic determinant in which all n sheets are equal, is $n!$ times the plane determinant formed from the elements in any one sheet, as they stand. (Dahlander.) Proof direct by development by columns, *e. g.*, by Sec. 2.

(6). The value of a cubic determinant in which, of the n sheets, the first q are equal to the first, while the remaining $(n-q)$ are equal to the $(q+1)$th identically, is $q!\,(n-q)!$ times the sum of the $n!\,/\,q!\,(n-q)!$ determinants formed by replacing in the 1st sheet, any $(n-q)$ columns (or rows) by the corresponding $(n-q)$ columns (or rows) of the $(q+1)$th sheet. (Scott, 1882.) Proof direct by development by columns (or rows) by Sec. 2. This theorem can easily be extended to the case where q_i sheets are equal to the ith, where $\Sigma\, q_i = n$.

(7). A cubic determinant in which any two sheets are composed of columns (or rows) such that every column (or every row) in each, is some multiple of a single column (or row) vanishes identically. For, each plane determinant in the development by columns (or rows) vanishes.

(8). Let us define the *principal diagonal* to be that diagonal of the cube which contains the elements *a_{11}, b_{22}, c_{33}, etc. Let us define the *diagonal plane*

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* See p. 51 for notation.

of rows (or *columns*, or *files*) to be that diagonal plane of the cube which contains the row (or column, or file) through a_{11}. And let us call the diagonal plane of files the *principal diagonal plane.*

If all the elements of any cubic determinant vanish, except those in the principal diagonal plane, the value of the determinant is equal to the value of the Scott's function formed of the elements in this plane, as they stand. (Scott, 1881.) Proof by development by files by Sec. 2.

Likewise, if all the elements are zero except those in the diagonal plane of columns (or rows) the determinant is equal to the ordinary determinant formed of the array of elements in that plane, as they stand. Proof by development by columns (or rows).

(9). If we denote as a *skew symmetric* cubic determinant, a determinant whose elements are such that all those in the principal diagonal plane vanish, while those on one side of it are got from those on the other by reflecting them in the principal diagonal plane and then changing their signs, a number of theorems for ordinary skew symmetric determinants may be generalized. The following will be typical:

A skew symmetric cubic determinant of odd order, vanishes. For, multiply each sheet through by -1. Then since n is odd, the determinant will change sign, by (4). But the effect of this operation is merely to interchange the sections and layers, and hence the value is unchanged, by (1). Hence that value must be zero. (Gegenbauer, 1883.)

(10). To form the derivative, with regard to a variable, of a cubic determinant of order n, whose elements are functions of that variable, write down the cubic determinant n times, differentiate every element in any one sheet (or layer, or section) in each determinant, not taking the same one twice, and the sum of the determinants so formed will be the required result. This can be proved at once by expanding by columns (or rows), and then differentiating by the ordinary rule.

(11). Let us now define the *minor* of any element of a cubic determinant to be the cubic determinant formed by striking out of the given determinant the three planes through this element. The *co-factor* of any element is the minor of that element taken with the sign plus or minus according as the sum of the two subscripts of the given element is even or odd. It is seen that the sign of the co-factor will be the same all along any file; and that the sign of the co-factor for the elements in any sheet is found precisely as if the array in that sheet were an ordinary determinant.

Any cubic determinant is equal to the sum of the products of the elements in any one plane multiplied by their respective co-factors. The proof is analogous to that for ordinary determinants, and depends upon the expansion of the given determinant into ordinary products. (Dahlander.) A proof by Sec. 2 is also easy.

(12). Hence, if two cubic determinants are identical except for the elements of some one given plane in each, their sum is the cubic determinant identical with either except for this same plane, any element of which is the sum of the corresponding elements in the two given determinants. (Dahlander.)

Conversely, if the elements of any plane of a cubic determinant are all polynomial, it may be written as the sum of several cubic determinants, in an obvious manner. (Dahlander.) The proof depends directly on (11).

(13). In any cubic determinant, we may add to any layer (or section; but *not* sheet) any multiple of any other layer (or section). For a determinant with two layers (or sections) one of which is a multiple of the other, vanishes, by (3) and (4). (Gegenbauer, 1881.)

(14). (*a*) A cubic determinant in which all of the elements in any one plane are zero except a single one, is equal to the product of that element and its co-factor. If all the elements in any plane are zero, the determinant vanishes. (Padova.) Proof by (11).

(*b*) Conversely, the order of any cubic determinant, by the reverse process to (*a*), may be raised by any number. (Padova.)

(*c*) A cubic determinant in which all the elements are zeros except those in the principal diagonal plane and those in some one sheet (or section, or layer), is equal to the Scott's function formed from the elements in the principal diagonal plane, as they stand. For the terms given by the Scott's functions evidently enter, by Sec. 2; and no others can enter, as is seen by forming the minor of any element in the sheet (or section, or layer) whose elements are arbitrary.

(15). Any cubic determinant of order n, is equal to the sum of the n cubic determinants formed by omitting from any one plane, in succession, all but a single line, no one being left twice and replacing the elements thus omitted by zeros. (Padova.) This is related to the theorem on expansion by minors. The proof is obvious either by (11), or by direct expansion by Sec. 2 into plane determinants (in case of rows or columns), or into Scott's functions (in case of files).

(16). *Laplace's Theorems* are easily extended to cubic determinants. Let us take any cubic determinant, and split it into two arrays: one of q sheets (or sections, or layers), and the other of $(n-q)$ sheets (or sections, or layers).

Out of the first array let us form a cubic determinant of order q by striking out of it any $(n-q)$ sections and any $(n-q)$ layers. Let us form the *conjugate determinant* from the second array, by striking out of it those q sections and those q layers *not* struck out of the first array, and prefixing to it the sign $+$ or $-$, according as the term which is the product of the elements in the principal diagonals of the two determinants is positive or negative in the complete expansion of the given determinant. *The given cubic determinant is equal to the sum of the products of every such pair of conjugate determinants that can be formed.* (Gegenbauer, 1881.) The proof consists in expanding the given cubic determinant, and each of those in the result, into plane determinants by Sec. 2, and then applying Laplace's theorem to each of the resulting plane determinants of order n.

In a manner analogous to (15) we have: Any cubic determinant is equal to the sum obtained by forming all possible cubic determinants by omitting from any q parallel planes, $(n-q)$ planes of either other sort, and adding the results. (Of course, the absent elements are supplied by zeros.)

Special and generalized forms of these theorems are obvious. From one such we see that: The product of any two cubic determinants of orders p and q is expressible as a cubic determinant of order $(p+q)$, whose principal diagonal consists of the principal diagonals of the two given determinants; the position of the other elements being obvious from (16). By aid of (8), (13) and the above, the theorem of the next article may be proved in a manner analogous to that used in ordinary determinants.

4. THE MULTIPLICATION THEOREM.

The product of any cubic determinant of order n and any plane determinant of order n, may be expressed as a cubic determinant of order n.* In forming this product determinant, we need merely replace each sheet of the given cubic determinant by a new sheet, the elements of which are formed from the array in the old sheet, and the array of the given plane determinant, in precisely the same way in which the elements in the product of two plane determinants are formed from the arrays which compose those determinants; the same order being observed in each sheet. (Padova.) There are then four distinct methods of multiplication, viz., those in which we combine: (1) *Columns* of the cubic determinant with *columns* of the plane determinant; (2) *Columns* (of the cubic determinant) with *rows* (of the plane determinant); (3) *Rows* with *columns*; (4) *Rows* with *rows*. We give the proof of (1). The proofs of (2), (3) and (4), are exactly like it. Expand the given cubic determinant into $n\,!$ plane determinants, by columns, by Sec. 2; multiply each of these by the given plane determinant by the ordinary method of columns and columns. The result will be $n\,!$ plane determinants of order n, in which there stand identical columns wherever there were such in the original expansion of the given cubic determinant. Hence this sum is itself a cubic determinant of order n, in which it is seen at once that the new elements are formed as stated in the above theorem.

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* It can be shown that the product of a determinant of *class* p (see the second foot-note in Sec. 2) and *order* n by a determinant of *class* q and *order* n is a determinant of *class* $(p+q-2)$ and *order* n. For a *cubic* determinant, $p=3$; for a *plane* determinant $q=2$; whence $(p+q-2)=3$, and the product is a *cubic* determinant.

There is no similar theorem in which the word *file* replaces the word *row* (or *column*), for there is no development by files in Sec. 2 into ordinary determinants.

It is now possible to multiply any cubic determinant by any plane determinant; for their orders may be made equal by (14), or by the similar rule in plane determinants.

5. INVARIANTS AND COVARIANTS.

(*a*) Let us consider n forms, $f_1, f_2, \ . \ . \ . \ , f_n$, each of n variables $x_1, x_2, \ . \ . \ ., x_n$; where the degree of each form is arbitrary. Let us *define* as the *Hessian*[*] *of these forms*, that cubic determinant of which the ith sheet is:

$$\boxed{\begin{array}{cccc}
\frac{\partial^2 f_i}{\partial x_1 \partial x_1}, & \frac{\partial^2 f_i}{\partial x_1 \partial x_2}, & \cdots\cdots, & \frac{\partial^2 f_i}{\partial x_1 \partial x_n} \\
\frac{\partial^2 f_i}{\partial x_2 \partial x_1}, & \frac{\partial^2 f_i}{\partial x_2 \partial x_2}, & \cdots\cdots, & \frac{\partial^2 f_i}{\partial x_2 \partial x_n} \\
\cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\
\frac{\partial^2 f_i}{\partial x_n \partial x_1}, & \frac{\partial^2 f_i}{\partial x_n \partial x_2}, & \cdots\cdots, & \frac{f\partial^2{}_i}{\partial x_n \partial x_n}
\end{array}} \ (i = 1, 2, \ . \ . \ . \ . \ , n). \qquad (1)$$

Then the Hessian of any such set of n forms in n variables, is a simul-

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[*] This nomenclature will cause no ambiguity. For when the number of forms is unity, the function reduces to a constant multiplied by the ordinary Hessian.

*taneous covariant of those forms.** (Gegenbauer, 1881.) To show this, let the Hessian be denoted by H. Let us perform upon the variables a linear transformation:

$$y_j = l_{1j}\,x_1 + l_{2j}\,x_2 + \cdot\cdot\cdot + l_{nj}\,x_n \qquad (j = 1, 2, \cdot\cdot\cdot\cdot, n) \tag{2}$$

whose determinant

$$D = \begin{vmatrix} l_{11}, l_{21}, & \cdot\cdot\cdot\cdot\cdot, & l_{n1} \\ l_{12}, l_{22}, & \cdot\cdot\cdot\cdot\cdot, & l_{n2} \\ \cdot\cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot \\ l_{1n}, l_{2n}, & \cdot\cdot\cdot\cdot\cdot, & l_{nn} \end{vmatrix} \tag{3}$$

does not vanish. Having introduced these variables into each of the given forms, let us form the new Hessian, H', from the transformed forms, now regarding the y's as variable, in place of the x's. We will show that

$$H \cdot D^2 \equiv H'. \tag{4}$$

By (2) we see that

$$\frac{\partial}{\partial y_j} = l_{1j}\frac{\partial}{\partial x_1} + l_{2j}\frac{\partial}{\partial x_2} + \cdot\cdot\cdot\cdot + l_{nj}\frac{\partial}{\partial x_n} \quad (j = 1, 2, \cdot\cdot\cdot, n). \tag{5}$$

Now by Sec. 4, the product HD is a cubic determinant whose ith sheet is

$$\boxed{\begin{array}{l} l_{11}\dfrac{\partial^2 f_i}{\partial x_1 \partial x_1} + l_{21}\dfrac{\partial^2 f_i}{\partial x_1 \partial x_2} + \cdots + l_{n1}\dfrac{\partial^2 f_i}{\partial x_1 \partial x_n}\,;\; l_{12}\dfrac{\partial^2 f_i}{\partial x_1 \partial x_1} + \cdots;\;\cdots\cdot;\; l_{1n}\dfrac{\partial^2 f_i}{\partial x_1 \partial x_1} + \cdots \\ l_{11}\dfrac{\partial^2 f_i}{\partial x_2 \partial x_1} + l_{21}\dfrac{\partial^2 f_i}{\partial x_2 \partial x_2} + \cdots + l_{n1}\dfrac{\partial^2 f_i}{\partial x_2 \partial x_n}\,;\; l_{12}\dfrac{\partial^2 f_i}{\partial x_2 \partial x_1} + \cdots;\;\cdots\cdot;\; l_{1n}\dfrac{\partial^2 f_i}{\partial x_2 \partial x_1} + \cdots \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ l_{11}\dfrac{\partial^2 f_i}{\partial x_n \partial x_1} + l_{21}\dfrac{\partial^2 f_i}{\partial x_n \partial x_2} + \cdots + l_{n1}\dfrac{\partial^2 f_i}{\partial x_n \partial x_n}\,;\; l_{12}\dfrac{\partial^2 f_i}{\partial x_n \partial x_1} + \cdots;\;\cdots\cdot;\; l_{1n}\dfrac{\partial^2 f_i}{\partial x_n \partial x_1} + \cdots \end{array}}$$

$$(i = 1, 2, \cdot\cdot\cdot, n)$$

the multiplication being performed by *rows* and *rows*. This may be written by (5) in the form:

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* The generalization of this and of the following theorems to m dimensions, is obvious and important. It is hoped that the attempt at simplicity in this paper will justify leaving these theorems to be inferred by analogy.

$$\begin{vmatrix} \frac{\partial}{\partial y_1}\left(\frac{\partial f_i}{\partial x_1}\right); & \frac{\partial}{\partial y_2}\left(\frac{\partial f_i}{\partial x_1}\right); & \cdots\cdots; & \frac{\partial}{\partial y_n}\left(\frac{\partial f_i}{\partial x_1}\right) \\ \frac{\partial}{\partial y_1}\left(\frac{\partial f_i}{\partial x_2}\right); & \frac{\partial}{\partial y_2}\left(\frac{\partial f_i}{\partial x_2}\right); & \cdots\cdots; & \frac{\partial}{\partial y_n}\left(\frac{\partial f_i}{\partial x_2}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial}{\partial y_1}\left(\frac{\partial f_i}{\partial x_n}\right); & \frac{\partial}{\partial y_2}\left(\frac{\partial f_i}{\partial x_n}\right); & \cdots\cdots; & \frac{\partial}{\partial y_n}\left(\frac{\partial f_i}{\partial x_n}\right) \end{vmatrix} \quad (i=1,\ 2,\ \ldots,\ n).$$

Now let us form $H\cdot D^2$, by multiplying the determinant just found by D, by Sec. 4, the multiplication being performed this time by *columns* and *rows*. By a reduction precisely similar to the above, the ith sheet becomes, by virtue of (5):

$$\begin{vmatrix} \frac{\partial^2 f_i}{\partial y_1 \partial y_1}; & \frac{\partial^2 f_i}{\partial y_1 \partial y_2}; & \frac{\partial^2 f_i}{\partial y_1 \partial y_3}; & \cdots\cdots; & \frac{\partial^2 f_i}{\partial y_1 \partial y_n} \\ \frac{\partial^2 f_i}{\partial y_2 \partial y_1}; & \frac{\partial^2 f_i}{\partial y_2 \partial y_2}; & \frac{\partial^2 f_i}{\partial y_2 \partial y_3}; & \cdots\cdots; & \frac{\partial^2 f_i}{\partial y_2 \partial y_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f_i}{\partial y_n \partial y_1}; & \frac{\partial^2 f_i}{\partial y_n \partial y_2}; & \cdots & \cdots\cdots; & \frac{\partial^2 f_i}{\partial y_n \partial y_n} \end{vmatrix} \quad (i=1,\ 2,\ \ldots,\ n).$$

But such a determinant is precisely H'. Hence (4) holds. Q. E. D.

Since by (3), Sec. 3, any two sheets (or any number of them) may be identical without causing the determinant to vanish, the functions $f_1, f_2, \ldots, f_n$ *need not be distinct*, but may have some duplicates among them.* In particular, *if all n forms coincide* their Hessian reduces to $n!$ times the ordinary Hessian of that form, by (5), Sec. 3.

If all the forms are quadratic forms, the form f_i may be written:

---

* When there are fewer than n forms, several simultaneous covariants of the forms can thus be written. It is also to be remembered that a covariant of a covariant is a covariant.

$$\begin{aligned} f_i \equiv \frac{1}{2} \Big\{ & h^{(i)}_{11}x_1^2 + h^{(i)}_{12}x_1x_2 + h^{(i)}_{13}x_1x_3 + \cdot\cdot\cdot\cdot + h^{(i)}_{1n}x_1x_n \\ & + h^{(i)}_{21}x_2x_1 + h^{(i)}_{22}x_2^2 + \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot + h^{(i)}_{2n}x_2x_n \\ & + \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\ & + h^{(i)}_{n1}x_nx_1 + h^{(i)}_{n2}x_nx_2 + \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot + h^{(i)}_{nn}x_n^2 \Big\}, \end{aligned}$$

where $h_{jk} = h_{kj}$.

The determinant H then has as its ith sheet the matrix of the coefficients $h^{(i)}$ in the form f_i. Hence the Hessian of n forms, each quadratic in n variables, is the cubic determinant formed of the matrix of the coefficients of the forms as they stand; and this is a simultaneous *invariant* of these given forms (Zehfuss.) As in the general case the forms need not be distinct, and when there are fewer than n *distinct* forms, there are several simultaneous invariants which can thus be formed.

A single illustration will be given.* Let

$$\begin{aligned} f \equiv\ & a_{11}x_1^2 + a_{12}x_1x_2 + \cdot\cdot\cdot + a_{14}x_1x_4 \\ & + a_{21}x_2x_1 + \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot + a_{24}x_2x_4 \\ & + \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot \\ & + a_{41}x_4x_1 + \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot + a_{44}x_4^2 = 0 \end{aligned}$$

be the equation of a quadric surface; and let

$$\phi \equiv lx_1 + mx_2 + nx_3 + px_4 = 0$$

be the equation of a plane. Then let us take f, f, f, and ϕ^2 as our four ($n = 4$) quadratic forms. It follows that:

$$\begin{array}{|l|l|l|llll|} a_{11}, & a_{12}, & a_{13}, & a_{14} & & & \\ a_{21}, & a_{11}, & a_{12}, & a_{13}, & a_{14} & & \\ a_{31}, & a_{21}, & a_{11}, & a_{12}, & a_{13}, & a_{14} & \\ a_{41}, & a_{31}, & a_{21}, & l^2, & lm, & ln, & lp \\ & x_{41}, & a_{31}, & ml, & m^2, & mn, & mp \\ & & a_{41}, & nl, & nm, & n^2, & np \\ & & & pl, & pm, & pn, & p^2 \end{array} \equiv H, \text{ (say)},$$

---

* The similar condition for a line tangent to a conic is readily obtainable.

is an invariant of f and ϕ. But, by a suitably chosen linear transformation, we can reduce f to a sum of squares: $f' \equiv a'_{11}x'^2_1 + a'_{22}x'^2_2 + a'_{33}x'^2_3 + a'_{44}x'^2_4 = 0$, and ϕ goes into $\phi' \equiv l'x'_1 + m'x'_2 + n'x'_3 + p'x'_4 = 0$. Whence H' is a cubic determinant in which the elements in the first three sheets are zeros, except the diagonals which are each $a'_{11}, a'_{22}, a'_{33}, a'_{44}$. Hence by (14) c, Sec. 3,

$$H' = 6 \left\{ l'^2 a'_{22}a'_{33}a'_{44} + m'^2 a'_{11}a'_{33}a'_{44} + n'^2 a'_{11}a'_{22}a'_{44} + p' \, a'_{11}a'_{22}a'_{33} \right\}.$$

But the vanishing of this expression is easily shown to be the necessary and sufficient condition that the plane $\phi' = 0$ be tangent to the surface $f' = 0$. Since H is invariant, it follows that $H = 0$ *is the necessary and sufficient condition that the given plane* $\phi = 0$ *be tangent to the given quadric* $f = 0$. This form of the condition is sometimes more convenient and symmetrical than that ordinarily given, to which it is of course equivalent. If l, m, n, p be regarded as the *plane* (or "*tangential*") *coordinates* of any plane in space, $H = 0$ is the equation of the quadric surface in plane coordinates. It is seen at once to be of the second degree.

The process here used is applicable generally; *i. e.*, in place of any of the forms f_i, we may put any power of itself, or more generally still, any form which is a function of all the given forms. The Hessian will still be a simultaneous covariant of the given set of forms.

(b) Another general set of covariants can be produced as follows: Let $f_1, f_2, \ldots, f_n$ be a set of n *distinct* forms in n variables $x_1, x_2, \ldots, x_n$. Let us form the cubic determinant whose ith sheet is:

$$\left| \begin{array}{l} \dfrac{\partial f_i}{\partial x_1}\dfrac{\partial f_i}{\partial x_1}; \ \dfrac{\partial f_i}{\partial x_1}\dfrac{\partial f_i}{\partial x_2}; \ \ldots\ldots; \ \dfrac{\partial f_i}{\partial x_1}\dfrac{\partial f_i}{\partial x_n} \\ \dfrac{\partial f_i}{\partial x_2}\dfrac{\partial f_i}{\partial x_1}; \ \dfrac{\partial f_i}{\partial x_2}\dfrac{\partial f_1}{\partial x_2}; \ \ldots\ldots; \ \dfrac{\partial f_i}{\partial x_2}\dfrac{\partial f_i}{\partial x_n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{\partial f_i}{\partial x_n}\dfrac{\partial f_i}{\partial x_1}; \ \dfrac{\partial f_i}{\partial x_n}\dfrac{\partial f_i}{\partial x_2}; \ \ldots\ldots; \ \dfrac{\partial f_i}{\partial x_n}\dfrac{\partial f_i}{\partial x_n} \end{array} \right| (i = 1, 2, \ldots, n).$$

The cubic determinant thus formed will be a simultaneous covariant of the given forms. (von Escherich.) The proof is practically a repetition of that given in (a), above, except that we have a product in place of a second derivative.

In case all of the forms f_i are linear, this invariant is evidently the same as the Hessian of the squares of the forms, except for the constant factor 2^n.

(c) It is at once apparent that we may combine (a) and (b). Given any set of forms $f_1, f_2, \ldots, f_n$, in n variables $x_1, x_2, \ldots, x_n$, let us form the cubic determinant by performing on any q of the forms $(0 \leqq q \leqq n)$ *the Hessian operation* (as I shall call the operation in (a)), to get q of the sheets of the cubic determinant; and then performing on the remaining $(n-q)$ forms *the Product operation* (as I shall call the operation in (b)), to get the other $(n-q)$ sheets of the cubic determinant.

The cubic determinant thus formed will be a simultaneous covariant of the given forms. The proof consists in a repetition of the proofs given for (a) and (b).

It is to be noted that the forms upon which the *Product operation* is performed must be *distinct* from each other, here and in (b), as else the cubic determinant vanishes by (7), Sec. 3.

The particular example under (a) can be regarded as formed by the Hessian operation on f, three times, followed by the Product operation on ϕ.

(d) Next, let any two sets of forms $f_1^{(1)}, f_2^{(1)}, \ldots, f_n^{(1)}$ and $f_1^{(2)}, f_2^{(2)}, \ldots, f_n^{(2)}$ be given, and let us form the cubic determinant whose ith sheet is

$$\left|\begin{array}{llll}
\frac{\partial f_i^{(1)}}{\partial x_1}\frac{\partial f_i^{(2)}}{\partial x_1}; & \frac{\partial f_i^{(1)}}{\partial x_1}\frac{\partial f_i^{(2)}}{\partial x_2}; & \cdots\cdots\cdots\cdots & ;\frac{\partial f_i^{(1)}}{\partial x_1}\frac{\partial f_i^{(2)}}{\partial x_n} \\
\frac{\partial f_i^{(1)}}{\partial x_2}\frac{\partial f_i^{(2)}}{\partial x_1}; & \frac{\partial f_i^{(1)}}{\partial x_2}\frac{\partial f_i^{(2)}}{\partial x_2}; & \cdots\cdots\cdots\cdots & ;\frac{\partial f_i^{(1)}}{\partial x_2}\frac{\partial f_i^{(2)}}{\partial x_n} \\
\cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\
\cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\
\frac{\partial f_i^{(1)}}{\partial x_n}\frac{\partial f_i^{(2)}}{\partial x_1}; & \frac{\partial f_i^{(1)}}{\partial x_n}\frac{\partial f_i^{(2)}}{\partial x_2}; & \cdots\cdots\cdots\cdots & ;\frac{\partial f_i^{(1)}}{\partial x_n}\frac{\partial f_i^{(2)}}{\partial x_n}
\end{array}\right| \quad (i=1, 2, 3, \cdots, n).$$

The cubic determinant thus formed will be a simultaneous covariant of the given forms. (von Escherich.) The proof is precisely as in (a), (b), or (c), except that we have to take into account now, that the layers and sections may be interchanged, by 1, Sec. 3. As in (b) and (c), we must avoid having two forms of the same set equal. But any number of forms of one set may be equal to forms in the other set. I will call this operation, the *Double Product operation.*

(*e*) As in (*c*) we may combine the operations of (*a*), (*b*), (*d*). Thus having given any set of forms $f_1, f_2, \ldots$ in n variables $x_1, x_2, \ldots, x_n$ we may form a cubic determinant of which q sheets $(0 \leqq q \leqq n)$ are formed by the Hessian operation performed on any q of the given forms; p sheets $(0 \leqq p \leqq n-q)$ by the Product operation performed on any p of the given forms; and the remaining r $(=n-p-q)$ sheets, by the Double Product operation performed on any two sets of r forms each, taken from the given forms; *and the cubic determinant thus formed will be a simultaneous covariant of the forms used.** The proof is the same as before. It is to be noticed that for particular values of p and q, this theorem covers all the previous cases, (*a*), (*b*), (*c*), (*d*). In forming the determinant, some of the forms may be identical and we need only be careful to make any of them distinct in case we desire to guard against the identical vanishing of the determinant, as in (*b*), (*c*), and (*d*).

6. ANALOGY TO ORDINARY PRODUCTS.

(1). *Notation.* Let us denote by $[a\, b\, c \ldots q]$ a cubic determinant of order n, in which the first sheet is:

$$\begin{array}{|l|}
\hline
a_{11},\ a_{12},\ a_{13},\ \ldots\ldots,\ a_{1n} \\
a_{21},\ a_{22},\ \ldots\ldots\ldots,\ a_{2n} \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
a_{n1},\ a_{n2},\ \ldots\ldots\ldots,\ a_{nn} \\
\hline
\end{array}$$

the successive sheets merely changing the *dominant letter*. There are always n dominant letters, and of each of these there are n^2 with different subscripts.

(2). *Some simple analogies.* In ordinary algebra we have: $ab \equiv ba$. Likewise here: $[ab] \equiv [ba]$ by (2), Sec. 3,

$$i.\,e.\quad \begin{array}{|c|c|}
\cline{1-1}
a_{11},\ a_{12} & \multicolumn{1}{c}{} \\ \cline{2-2}
a_{21}, & b_{11},\ b_{12} \\ \cline{1-1}
\multicolumn{1}{c|}{} & b_{21},\ b_{22} \\ \cline{2-2}
\end{array}
\quad\equiv\quad
\begin{array}{|c|c|}
\cline{1-1}
b_{11},\ b_{12} & \multicolumn{1}{c}{} \\ \cline{2-2}
b_{21}, & a_{11},\ a_{12} \\ \cline{1-1}
\multicolumn{1}{c|}{} & a_{21},\ a_{22} \\ \cline{2-2}
\end{array}\ .$$

---

* The generalization of this theorem to m dimensions gives a very general theorem. It is believed that this is obvious without the rather complex statement of it.

In ordinary algebra: $(a+b)c \equiv ac + bc.$
Likewise here: $[(a+b)c] \equiv [ac] + [bc]$ by (12), Sec. 3,

$$i.\ e.\quad \begin{array}{|l} a_{11}+b_{11},\ a_{12}+b_{12} \\ a_{21}+b_{21}, \begin{array}{|cc|}\hline c_{11}, & c_{12} \\ c_{21}, & c_{22} \\ \hline \end{array} \end{array} \equiv \begin{array}{|l} a_{11},\ a_{12} \\ a_{21}, \begin{array}{|cc|}\hline c_{11}, & c_{12} \\ c_{21}, & c_{22} \\ \hline \end{array} \end{array} + \begin{array}{|l} b_{11},\ b_{12} \\ b_{21}, \begin{array}{|cc|}\hline c_{11}, & c_{12} \\ c_{21}, & c_{22} \\ \hline \end{array} \end{array}$$

Laplace's theorem (see (16), Sec. 3) may evidently be stated for $n = 4$ as follows:

$$\sum [abcd] \equiv \sum [ab]\cdot[cd]$$

where Σ indicates the sum of all possible combinations such as that indicated. In the use of such forms, however, considerable care is necessary, and they are avoided in what follows.

Any simple homogeneous algebraic formula is seen at once to have an analogue in cubic determinants.

Take $a^2 + 2ab + b^2 \equiv (a+b)^2.$
We have $[aa] + 2\,[ab] + [bb] \equiv [(a+b)(a+b)],$

$$i.e.\ \begin{array}{|l} a_{11},\ a_{12} \\ a_{21}, \begin{array}{|cc|}\hline a_{11}, & a_{12} \\ a_{21}, & a_{22} \\ \hline \end{array} \end{array} + 2 \begin{array}{|l} a_{11},\ a_{12} \\ a_{21}, \begin{array}{|cc|}\hline b_{11}, & b_{12} \\ b_{21}, & b_{22} \\ \hline \end{array} \end{array} + \begin{array}{|l} b_{11},\ b_{12} \\ b_{21}, \begin{array}{|cc|}\hline b_{11}, & b_{12} \\ b_{21}, & b_{22} \\ \hline \end{array} \end{array} \equiv \begin{array}{|l} a_{11}+b_{11}, a_{12}+b_{12} \\ a_{21}+b_{21}, \begin{array}{|cc|}\hline a_{11}+b_{11}, & a_{12}+b_{12} \\ a_{21}+b_{21}, & a_{22}+b_{22} \\ \hline \end{array} \end{array}.$$

It is seen by (5), Sec. 3, that any square term gives a cubic determinant which degenerates into a plane determinant. Thus the above becomes:

$$\begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} + \begin{array}{|l} a_{11},\ a_{12} \\ a_{21}, \begin{array}{|cc|}\hline b_{11}, & b_{12} \\ b_{21}, & b_{22} \\ \hline \end{array} \end{array} + \begin{vmatrix} b_{11}, & b_{12} \\ b_{21}, & b_{22} \end{vmatrix} \equiv \begin{vmatrix} a_{11}+b_{11}, & a_{12}+b_{12} \\ a_{21}+b_{21}, & a_{22}+b_{22} \end{vmatrix}.$$

Likewise $[aa] - [bb] \equiv [(a-b)(a+b)].$ And so on.

(3) In fact we may state at once a *Law of Analogy*:

To any homogeneous formula in ordinary algebra there corresponds a *theorem in cubic determinants.* This follows from the fact that we may start with the formulæ

$$[ab] = [ba]$$

and

$$[(a+b)c] = [ac] + [bc]$$

and build up the cubic determinant theorem by methods exactly analogous to those used in ordinary algebra.

(4) *Examples.* Several examples follow, in each of which a direct verification is easy.

$$[\,(x+y+z)^3] - [x^3] - [y^3] - [z^3] \equiv 3\,[\,(y+z)\;(z+x)\;(x+y)\,].$$

Since each cube gives a degenerating cubic determinant, we get, by (5), Sec. 3

$$2\begin{vmatrix} x_{11}+y_{11}+z_{11}\,; & x_{12}+y_{12}+z_{12}\,; & x_{13}+y_{13}+z_{13} \\ x_{21}+y_{21}+z_{21}\,; & x_{22}+y_{22}+z_{22}\,; & x_{23}+y_{23}+z_{23} \\ x_{31}+y_{31}+z_{31}\,; & x_{32}+y_{32}+z_{32}\,; & x_{33}+y_{33}+z_{33} \end{vmatrix} - 2\begin{vmatrix} x_{11}\,; & x_{12}\,; & x_{13} \\ x_{21}\,; & x_{22}\,; & x_{23} \\ x_{31}\,; & x_{32}\,; & x_{33} \end{vmatrix} - 2\begin{vmatrix} y_{11}\,; & y_{12}\,; & y_{13} \\ y_{21}\,; & y_{22}\,; & y_{23} \\ y_{31}\,; & y_{32}\,; & y_{33} \end{vmatrix}$$

$$-2\begin{vmatrix} z_{11}\,; & z_{12}\,; & z_{13} \\ z_{21}\,; & z_{22}\,; & z_{23} \\ z_{31}\,; & z_{32}\,; & z_{33} \end{vmatrix} \equiv \begin{array}{|lllll} y_{11}+z_{11}\,; & y_{12}+z_{12}\,; & y_{13}+z_{13} & & \\ y_{21}+z_{21}\,; & z_{11}+x_{11}\,; & z_{12}+x_{12}\,; & z_{13}+x_{13} & \\ y_{31}+z_{31}\,; & z_{21}+x_{21}\,; & x_{11}+y_{11}\,; & x_{12}+y_{12}\,; & x_{13}+y_{13} \\ & z_{31}+x_{31}\,; & x_{21}+y_{21}\,; & x_{22}+y_{22}\,; & x_{23}+y_{23} \\ & & x_{31}+y_{31}\,; & x_{32}+y_{32}\,; & x_{33}+y_{33} \end{array}\;;$$

a rather interesting result, especially for some special values of the (x, y, z)'s, and it is easily verified.

The binomial theorem is interesting. We find:

$$[(a+b)^n] \equiv [a^n] + n\,[a^{n-1}b] + \frac{n(n-1)}{2\,!}\,[a^{n-2}b^2] + \cdots\cdot$$

$$+\,\frac{n(n-1)\cdot\cdot\cdot\cdot\cdot(n-r+1)}{r\,!}\,[a^{n-r}b^r] + \cdots\cdot$$

$$+\,\frac{n(n-1)}{2\,!}\,[a^2b^{n-2}] + n\,[ab^{n-1}] + [b^n],$$

where n is of course integral. Whence by (5) and (6), Sec. 3, if we denote by $S^{(j)}_{(n-j)}$ the sum of the $\dfrac{n\,!}{(n-j)\,!\,j\,!}$ plane determinants which can be formed by substituting any j rows of the plane determinant of the matrix of the b's, for the corresponding j rows of the plane determinant of the matrix of the a's; and if we denote by D the determinant of the matrix of the $(a+b)$'s, then:

$$n\,!D \equiv n\,!\,(S^{(0)}_{(n)}) + n\Big((n-1)\,!\cdot 1\,!\,S^{(1)}_{(n-1)}\Big) + \frac{n(n-1)}{2\,!}\Big((n-2)\,!\,2\,!\,S^{(2)}_{(n-2)}\Big) + \cdots$$
$$\cdots\cdots + \frac{n(n-1)\cdots(n-r+1)}{r\,!}\Big((n-r)\,!\,r\,!\,S^{(r)}_{(n-r)}\Big) + \cdots + n\,!(S^{(n)}_{(0)}),$$

which reduces at once to

$$D \equiv S^{(0)}_{(n)} + S^{(1)}_{(n-1)} + \cdots\cdots + S^{(r)}_{(n-r)} + \cdots\cdots + S^{(n-1)}_{(1)} + S^{(n)}_{(0)}\,;$$

a very peculiar result, which can be verified immediately.

Similarly:

$$\left[\prod_{i=1}^{i=n}(a^{(i)}+b^{(i)})\right] = \left[\prod_{i=1}^{i=n}(a^{(i)})\right] + \sum_{j=1}^{j=n}\left[\prod_{j\neq i=1}^{j\neq i=n-1}(a^{(i)}b^{(j)})\right] + \cdots$$
$$\cdots + \Sigma\left[\prod_{i=1}^{i=n-r}a^{(i)}\prod_{j=n-r}^{j=n}b^{(j)}\right] + \cdots\cdots + \left[\prod_{i=1}^{i=n}(b^{(i)})\right],$$

where, as usual, Π denotes a product and Σ a summation. This of course includes the last example. It also includes the case where $a^{(i)} = a^{(1)}$ for all values of i. This gives:

$$\left[\prod_{i=1}^{i=n}(a+b^{(i)})\right] = \left[a^n\right] + \sum_{j=1}^{j=n}\left[a^{n-1}b^{(j)}\right] + \cdots + \Sigma\left[a^{n-r}\prod_{j=1}^{j=r}(b^{(j)})\right]$$
$$+ \cdots\cdots + \Sigma\left[a\prod_{j=1}^{j=n-1}(b^{(j)})\right] + \left[\prod_{i=1}^{i=n}(b^{(i)})\right].$$

Let us now specialize the b's.* Let the cubical determinant $D \equiv \left[\prod_{i=1}^{i=n}(a+b^{(i)})\right]$ have each plane identical with the matrix of a's, except that in the ith sheet, the ith column (or row) is multiplied by $(1+h)$. Then the cubic determinant $\left[\prod_{i=1}^{i=n}(b^{(i)})\right]$ will have all its elements equal to zero, except that in the ith sheet, the ith row is h times the corresponding row in $[a^n] \equiv D'$ (say). We find at once by (5) and (6), Sec. 3:

$$D \equiv D'\left\{1 + \frac{1}{(n-1)\,!\,1\,!}(n-1)\,!\,h + \cdots + \frac{1}{(n-r)\,!\,r\,!}(n-r\,!)h^r + \cdots + \frac{1}{n\,!}h^n\right\}$$
$$\equiv D'\left\{1 + \frac{h}{1\,!} + \frac{h^2}{2\,!} + \cdots + \frac{h^r}{r\,!} + \cdots + \frac{h^n}{n\,!}\right\}.$$

---

* I owe this particular example to F. R. Scott (*Proc. Lond. Math. Soc.*, XIII), whose proof, of course, differs from mine.

Whence

$$\lim_{n=\infty}\left(\frac{D}{D'}\right)=e^h;$$

another peculiar result.

7. EXTENSION TO *m* DIMENSIONS.

It is to be noticed that in extending these theorems to a determinant of class m (*i. e.* in m dimensions) no difficulty is experienced since our arrays do not really depend upon the geometrical form which we have so far given them. But the use of such determinants becomes more complicated as the possibility of geometrical representation of the matrix is lost.

The theorems themselves are easily suggested by those here given, and by the ordinary theorems in plane determinants. The proofs are usually easy, either directly, in the case of any particular dimension, or by a process of mathematical induction, in general, the determinant of class m being defined, as in the second foot-note in Sec. 2, as the sum of $n!$ determinants of class $(m-1)$.

It is noticed immediately that any determinant whose class is even is symmetrical with respect to all $2p$ dimensions. But a determinant of odd class (*e. g.* a cubic determinant) is unsymmetrical with respect to one dimension, but symmetrical with respect to the remaining $2p$ dimensions.

A determinant whose class is unity is evidently an ordinary product. It is interesting to notice that all general rules for m dimensional determinants apply to products when $m = 1$. This in fact accounts for the analogy between cubic determinants and ordinary products, mentioned in Sec. 6, and formed the basis of a great part of the work in this paper.

We see at once that this same relation of analogy holds for any odd-dimensional determinant, for similar reasons. It is, perhaps, less interesting to notice that a similar relation of analogy holds between plane determinants (or in fact any determinants of even class), and the algebra of "alternate numbers," these alternate numbers having been defined, by Clifford, exactly so as to satisfy the laws of ordinary determinants.*

CAMBRIDGE, MASS., APRIL, 1899.

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* Compare this fact with the use made of alternate numbers by F. R. Scott, *Theory of Determinants.*